PERTURBATION THEORY

Time-independent

1. Using first-order perturbation theory, calculate the energy of the nth excited state for a spinless particle of mass m moving in an infinite potential well of length $2L$, with walls at $x = 0$ and $x = 2L$, i.e.,

$$V(x) = \begin{cases} 0, & 0 \le x \le 2L, \\ \infty, & \text{otherwise} \end{cases}, \tag{1}$$

which is modified at the bottom by the following two perturbations:

a) $V_p(x) = \lambda V_0 \sin[\pi x/(2L)]$. [Sol: $E_n = \frac{\hbar^2\pi^2}{8mL^2}n^2 + \frac{2\lambda V_0}{\pi}\frac{4n^2}{4n^2-1}$].

b) $V_p(x) = \lambda V_0 \delta(x-L)$, [Sol: $E_n = \frac{\hbar^2\pi^2}{8mL^2}n^2 + a_n \lambda V_0/L$, $a_n = 0$ if n is even and $a_n = 1$ if n is odd].

where $\lambda << 1$. The exact expressions of the energy levels and of the wave functions for the potential $V(x)$ are

$$E_n = \frac{\hbar^2\pi^2}{8mL^2}n^2, \quad \psi_n(x) = \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi}{2L}x\right), \quad n = 1, 2\ldots \tag{2}$$

2. Consider a system whose Hamiltonian is given by

$$H = E_0 \begin{pmatrix} 1+\lambda & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 3 & -2\lambda \\ 0 & 0 & -2\lambda & 7 \end{pmatrix}, \tag{3}$$

where $\lambda << 1$.

a) By decomposing this Hamiltonian into $H = H_0 + H_p$, find the eigenvalues and eigenstates of the unperturbed Hamiltonian H_0. [Sol: Eigenenergies: E_0, $8E_0$, $3E_0$ and $7E_0$; Corresponding eigenvectors: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$].

b) Diagonalize H to find the exact eigenvalues of H; expand each eigenvalue to the second power of λ. [Sol: Eigenenergies: $(1+\lambda)E_0$, $8E_0$, $(5-2\sqrt{1+\lambda^2})E_0 \simeq (3-\lambda^2)E_0$, $(5+2\sqrt{1+\lambda^2})E_0 \simeq (7+\lambda^2)E_0$]

c) Using first- and second-order nondegenerate perturbation theory, find the approximate eigenenergies of H and the eigenstates to first order. Compute these with the exact values obtained in b). [Sol: Eigenenergies: $(1+\lambda)E_0$, $8E_0$, $(3-\lambda^2)E_0$, $(7+\lambda^2)E_0$; Corresponding eigenvectors: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\\lambda/2\end{pmatrix}, \begin{pmatrix}0\\0\\-\lambda/2\\1\end{pmatrix}$]

3. Consider a perturbation $H_p = bx^4$ to the simple harmonic oscillator Hamiltonian $H_0 = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$. This is an example of an anharmonic oscillator, one with a nonlinear restoring force.

   a) Show that the first-order shift in the energy is given by $E_n^{(1)} = \frac{3\hbar^2 b}{4m^2\omega^2}(1+2n+2n^2)$.

   b) Argue that no matter how small b is, the perturbation expansion will break down for some sufficiently large n. What is the physical reason?

4. The spin Hamiltonian of a spin 1 ion in a crystal is given by $H = \frac{a}{\hbar^2}S_z^2 + \frac{b}{\hbar^2}(S_x^2 - S_y^2)$. Assume $b << a$ and treat $\frac{b}{\hbar}(S_x^2 - S_y^2)$ as a perturbation. Calculate the unperturbed eigenenergies and the first-order corrections using perturbation theory. Beware of the degeneracy. Compare your perturbative results with the exact eigenvalues. [Sol: Exact eigenenergies: $a \pm b$ and 0; Perturbative solution: eigenvalues of H_0 are a (twofold degeneracy) and 0 (no degeneracy); full solution coincides with the exact one].

5. Consider an isotropic harmonic oscillator in two dimensions. The Hamiltonian is $H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2)$.

   a) What are the energies of the three lowest-lying states? Is there any degeneracy? [Sol: Eigenvectors: $|0,0\rangle$, $|1,0\rangle$ and $|0,1\rangle$; Corresponding eigenenergies: $\hbar\omega$, $2\hbar\omega$, $2\hbar\omega$; The first excited state is double degenerate].

   b) We now apply a perturbation $H_p = \beta m\omega^2 xy$, where β is a dimensionless real number much smaller than unity. Find the zeroth-order energy eigenket and the corresponding energy to first order for each of the three lowest-lying states. [Sol: Eigenvectors: $|0,0\rangle$, $(|1,0\rangle \pm |0,1\rangle)/\sqrt{2}$; Corresponding eigenenergies: $\hbar\omega$, $(2 \pm \beta/2)\hbar\omega$].

c) Solve the $H_0 + H_p$ problem exactly. Compare with the perturbation results obtained in *b*). [Sol: Same eigenvectors; Corresponding eigenenergies: $(\hbar\omega/2)[\sqrt{1+\beta} + \sqrt{1-\beta}] \simeq \hbar\omega$, $\hbar\omega[(3/2)\sqrt{1+\beta} + (1/2)\sqrt{1-\beta}] \simeq (2+\beta/2)\hbar\omega$, $\hbar\omega[(1/2)\sqrt{1+\beta} + (3/2)\sqrt{1-\beta}] \simeq (2-\beta/2)\hbar\omega$].

Hint: The problem separates into independent x and y harmonic oscillators with states $|n_x, n_y\rangle$; Item c) You might need to introduce new variables $x' = x'(x,y)$ and $y' = y'(x,y)$ to find the exact solution.

Variational method

6. Use the variational method to estimate the energy of the ground state of a one-dimensional harmonic oscillator by making use of the following two trial functions: (a) $\psi_0(x,\alpha) = Ae^{-\alpha|x|}$, (b) $\psi_0(x,\alpha) = A/(x^2+\alpha)$, where α is a positive real number and where A is the normalization constant. [Sol: $\hbar\omega/\sqrt{2}$ in both items].

7. For a particle of mass m moving in a one-dimensional box with walls at $x = 0$ and $x = L$, use the variational method to estimate:

   a) Its ground state energy. [Sol: $10\hbar^2/(2mL^2)$].

   b) Its first excited state energy. [Sol: $42\hbar^2/(2mL^2)$].

   Hint: The exact eigenenergies are known to be $E_n = \pi^2\hbar^2 n^2/(2mL^2)$. Ground state wave function: It has no nodes and is symmetric (even) with respect $x = L/2$; First excited state wave function: It has one node at $x = L/2$ and is odd with respect to $x = L/2$.

8. Consider a particle of mass m that is bouncing vertically and elastically on a reflecting hard floor where

$$V(z) = \begin{cases} mgz, & z > 0 \\ +\infty, & z \le 0 \end{cases}, \tag{4}$$

   and g is the gravitational constant. Use the variational method to estimate the ground state energy of this particle. The exact ground state energy is given by $2.338\left(\frac{1}{2}mg^2\hbar^2\right)^{1/3}$. [Sol: $\frac{3}{2}\left(\frac{9}{2}\right)^{1/3}\left(\frac{1}{2}mg^2\hbar^2\right)^{1/3}$].

Time-dependent

9. Consider a one-dimensional simple harmonic oscillator whose classical angular frequency is ω_0. For $t < 0$ it is known to be in the ground state. For $t > 0$ there is also a time-dependent potential $V(t) = F_0 x\cos(\omega t)$, where F_0 is a constant in both space and time. Obtain an expression for the expectation value $\langle x\rangle$ as a function of time using time-dependent perturbation to lowest nonvanishing order. Is this procedure valid for $\omega \simeq \omega_0$? $\left[\text{Sol: } -\frac{F_0}{m}\frac{\cos(\omega t)-\cos(\omega_0 t)}{\omega_0^2-\omega^2}\right]$.

10. A one-dimensional harmonic oscillator of frequency ω_0 is in its ground state for $t < 0$. For $t \geq 0$ it is subjected to a time-dependent but spatially uniform force in the x-direction, $\boldsymbol{F}(t) = F(t)\boldsymbol{i}$, with $F(t) = F_0 e^{-t/\tau}$.

    a) Using time-dependent perturbation theory to first order, obtain the probability of finding the oscillator in its first excited state for $t > 0$. Show that the $t \to \infty$ (τ finite) limit of your expression is independent of time. Is this reasonable or surprising? $\left[\text{Sol: } \frac{F_0^2}{2m\omega_0}\frac{e^{-2t/\tau}-2e^{-t/\tau}\cos(\omega_0 t)+1}{\omega_0^2+1/\tau^2} \to \frac{1}{2m\omega_0}\frac{F_0^2\tau^2}{1+\omega_0^2\tau^2} \text{ for } t\to\infty\right]$

    b) Can we find higher excited states? [Sol: not in first order].

11. A particle, initially ($i.e$, $t \to -\infty$) in its ground state in an infinite potential well whose walls are located at $x = 0$ and $x = a$, is subject at time $t = 0$ to a time-dependent perturbation $V(t) = \epsilon x e^{-t^2}$, where ϵ is a small real number. Calculate the probability that the particle will be found in its first excited state after a sufficiently long time, i.e., $t \to \infty$. The eigenenergy and wave function of the particle in the nth excited state in the infinite potential well are given by $E_n = n^2\pi^2\hbar^2/(2ma^2)$ and $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$, $n = 1, 2, \ldots$, respectively. $\left[\text{Sol: } \pi\left(\frac{16\epsilon a}{9\pi^2\hbar}\right)^2 \exp\left(-\frac{9\pi^4\hbar^2}{8m^2a^4}\right)\right]$.

POTENTIAL SCATTERING

Basic Notions

12. A proton beam producing a current of 5×10^{-9} amps (amps=Amperes) is incident on a target of copper. Assume the target thickness is such that the areal density is $\rho = 0.2\,\text{mg/cm}^2$. The detector has an area of 0.5 cm^2, normal to the scattered beam, and is 20 cm from the target.

If 10 protons are counted by the detector every second at a particular angle, calculate the differential cross section for protons scattering off copper at that angle. Hints: the electric charge carried by a proton, e, is 1.602×10^{-19} C. The mass of copper is 63.546 u. [Sol:~ 0.14 b/steradian].

The stationary-state approach

The integral equation of potential scattering

13. By introducing the Green's functions

$$G_\epsilon^\pm(\boldsymbol{x}') = \frac{1}{4\pi^2 i r} \int_{-\infty}^{+\infty} d\kappa \frac{\kappa e^{i\kappa r}}{\kappa^2 - (k^2 \pm i\epsilon)}, \tag{5}$$

where $r = |\boldsymbol{x}'|$, $\epsilon << k^2$ (with $\epsilon > 0$), show that

$$G^\pm(\boldsymbol{x}') = \frac{1}{4\pi} \frac{e^{\pm ikr}}{r} = \lim_{\epsilon \to 0} G_\epsilon^\pm(\boldsymbol{x}'). \tag{6}$$

The Born approximation

14. a) Calculate the differential cross section in the Born approximation for elastic scattering of a spinless particle of mass m and linear momentum $\hbar\boldsymbol{k}$ under the influence of the potential $V(r) = V_0 \frac{e^{-r/R}}{r}$, known as the *Yukawa potential.* [Sol: $(4m^2V_0^2/\hbar^4)/[1/R^2+4k^2\sin^2(\theta/2)]^2$, $k = |\boldsymbol{k}|$].

    b) Calculate the total cross section. [Sol: $(16\pi m^2 V_0^2 R^4/\hbar^4)/[1+8mER^2/\hbar^2]$.]

    c) Find the relation between V_0 and R so that the Born approximation is valid. [Sol: $[mV_0/(\hbar^2 k)][(1/4)[\ln(1+4k^2R^2)]^2+(\tan^{-1}(2kR))^2]^{1/2} << 1$]

15. Find an expression, in the Born approximation, for the scattering amplitude due to elastic scattering of a spinless particle of mass m and linear momentum $\hbar\boldsymbol{k}$ off a potential which is a superposition of Yukawa potentials, i.e., $V(r) = \int_M^\infty d\mu\sigma(\mu)\frac{e^{-\mu r}}{r}$. [Sol: $f(\theta) = -(2m/\hbar^2)\int_M^\infty d\mu\sigma(\mu)/(K^2+\mu^2)$, where $K^2 = 4k^2\sin^2(\theta/2)$, $k = |\boldsymbol{k}|$].

16. Consider a spinless particle of mass m and linear momentum $\hbar\boldsymbol{k}$ under the influence of a potential V.

a) Study the possibility of applying the Born approximation to the spherical potential well $V(r) = \begin{cases} -V_0, & r < a, \\ 0, & r \geq a \end{cases}$, where $V_0 > 0$, $a > 0$. [Sol: $(mV_0/(k^2\hbar^2))\sqrt{\sin^2(ka) + ka\{ka - \sin(2ka)\}} << 1$, $k = |\boldsymbol{k}|$].

b) Calculate the differential cross section in the Born approximation. [Sol: $(2mV_0^2a^2/(K^4\hbar^4))[\cos(Ka) - \sin(Ka)/(Ka)]^2$; $K = 2k\sin(\theta/2)$, with θ being the scattering angle of the particle].

c) Repeat item b) for the Gaussian potential $V(r) = V_0 e^{-\mu r^2}$, $\mu > 0$. [Sol: $(\pi m^2 V_0^2/(4\hbar^4\mu^3))e^{-K^2/(2\mu)}$, $K = 2k\sin(\theta/2)$].

17. Find the differential and total cross sections in the Born approximation for the elastic scattering of a spinless particle of mass m, linear momentum $\hbar\boldsymbol{k}$, which is initially traveling along the z-axis, from a nonspherical, double-delta potential $V(\boldsymbol{x}') = V_0\delta(\boldsymbol{x}' - a\hat{\boldsymbol{z}}) + V_0\delta(\boldsymbol{x}' + a\hat{\boldsymbol{z}})$, $V_0 > 0$, where $\hat{\boldsymbol{z}}$ is the unit vector along the z-axis.
[Sol: $(m^2V_0/(\pi^2\hbar^4))\cos^2(2ak\sin^2(\theta/2))$, $m^2V_0(4ak + \sin(4ak))/(4\pi ak\hbar^4)$, where $k = |\boldsymbol{k}|$ and θ is the scattering angle of the particle].

The method of partial waves

Partial-wave analysis

18. By using the corresponding scattering amplitude in the Born approximation, compute the s-wave phase shift ($l = 0$) for a spinless particle of mass m and linear momentum $\hbar\boldsymbol{k}$ under the influence of the potential

a) $V(r) = V_0 e^{-\mu r^2}$, $\mu > 0$. [Sol: $(1/(2i))\ln[1 - i(mV_0/(\hbar^2k))\sqrt{\pi/\mu}(1 - \exp(-k^2/\mu))]$, $k = |\boldsymbol{k}|$].

b) $V(r) = \begin{cases} -V_0, & r < a, \\ 0, & r \geq a \end{cases}$, where $V_0 > 0$. [Sol: $(2mV_0a^2(ka)/\hbar^2)(1 - \sin(2ka)/(2ka))$].

19. By considering a central potential and the asymptotic form

$$\psi_{\boldsymbol{k}}^{+}(\boldsymbol{x}') \sim \sum_{l=0}^{\infty} a_l P_l(\cos\theta)\frac{\sin(kr - l\pi/2 + \delta_l)}{kr} = e^{ikz'} + \frac{e^{ikr}}{r}f(\theta),$$

for the scattered wave function of a spinless particle of mass m, linear momentum $\hbar\boldsymbol{k}$ ($k = |\boldsymbol{k}|$), at the detection point $\boldsymbol{x}'$ [with spherical coor-

dinates (r,θ,ϕ)], and

$$f(\theta) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} d^3x'' e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}''} U(\boldsymbol{x}'')\psi^+_{\boldsymbol{k}}(x''), \tag{7}$$

with (r',θ',ϕ') being the spherical coordinates for $\boldsymbol{x}''$ and

$$\psi^+_{\boldsymbol{k}}(\boldsymbol{x}'') = \sum_{l=0}^{\infty} a_l P_l(\cos\theta')\frac{u_{l,k}(r')}{r'},$$

calculate the coefficients a_l and show that $a_l = i^l(2l+1)e^{i\delta_l}$.

Phase shifts, cross sections and optical theorem

20. Compute the phase shifts for elastic scattering of a particle of mass m and linear momentum $\hbar\boldsymbol{k}$ by a potential $V(r) = V_0 a\delta(r-a)$. Hints: The radial part of the wave function for such potential is given by $R_{l,k}(r) = \begin{cases} B_l j_l(kr), & r<a, \\ A_l[\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr)], & r>a, \end{cases}$, where $k = |\boldsymbol{k}|$, B_l and A_l are normalization constants. This wave function is continuous at $r = a$, but its derivative is discontinuous. Integrating the radial Schrödinger equation about $r = a$, $\frac{1}{R_{l,k}(a+0)}\frac{dR_{l,k}(a+0)}{dr} - \frac{1}{R_{l,k}(a-0)}\frac{dR_{l,k}(a-0)}{dr} = \frac{2mV_0 a}{\hbar^2}$. [Sol: $\tan\delta_l = \frac{(2mV_0a^2/\hbar^2)j_l^2(ka)}{(2mV_0a^2/\hbar^2)j_l(ka)n_l(ka)-1/(ka)}$].

21. Consider the elastic scattering of a spinless particle of mass m from a hard sphere potential: $V(r) = \infty$ for $r < a$ and $V(r) = 0$ for $r > a$.

    a) Calculate the total cross section in the low-energy limit. Find a numerical estimate for the cross section for the case of scattering 5 keV protons from a hard sphere of radius $a = 6$ fm. [Sol: $\simeq 4\pi a^2$, 4.5 barn]

    b) Calculate the total cross section in the high-energy limit. Find a numerical estimate for the cross section for the case of 700 MeV protons with $a = 6$ fm. [Sol: $\simeq 2\pi a^2$, 2.26 barn].

    Hints: For $l = 0$, the solution of the radial Schrödinger equation for a hard sphere potential is given by $u(r) = 0$ for $r < a$ and $u(r) = A\sin(kr + \delta_0)$ for $r > a$, with A being a normalization constant and $k = |\boldsymbol{k}|$. For a proton of mass m_p, $m_p c^2 = 938.27$ MeV and $\hbar c = 197.33$ MeV$\cdot$ fm. You may need to use that $\sum_{l=0}^{n}(2l+1) = (n+1)^2$.

22. Consider elastic scattering of 50 MeV neutrons from a nucleus. The phase shifts measured in this experiment are $\delta_0 = 95°$, $\delta_1 = 72°$, $\delta_2 = 60°$, $\delta_3 = 35°$, $\delta_4 = 18°$, $\delta_5 = 5°$; all other phase shifts are negligible.

    a) Find the total cross section. [Sol: 0.558 barn]

    b) Estimate the radius of the nucleus. [Sol: 4.17 fm].

    Use that for a neutron of mass m_n, $m_n c^2 = 939.57$ MeV; $\hbar c = 197.33$ MeV·fm.

23. Consider the elastic scattering of a spinless particle of mass m and linear momentum $\hbar\boldsymbol{k}$ from a spherically symmetric potential. If $d\sigma/d\Omega$ is the differential cross section and σ is the total cross section, show that

$$\sigma \leq \frac{4\pi}{k}\sqrt{\left.\frac{d\sigma}{d\Omega}\right|_{\theta=0}}.$$

    Verify this inequality explicitly for a general central potential using the partial-wave expansion of the scattering amplitude and the cross section.

Scattering length and effective range

24. a) Given the potential $V(r) = \begin{cases} -V_0, & r < a, \\ 0, & r \geq a \end{cases}$, where $V_0 > 0$, find the phase shift, effective range and the scattering length for the s-wave ($l = 0$). Assume that $V_0 << 1$, i.e., we have a weak potential producing $|\delta_l| << 1$ [Sol: $\sin\delta_0 \simeq -(C/3)[ka - (ka)^3/5]$, $Ca/3$, $[C/3 - 6/(5C)]a$, with $C = 2mV_0a^2/\hbar^2$].

    b) Repeat item a) for the Yukawa potential $V = -V_0 e^{-\mu r}/r$, with $V_0 > 0$, $\mu > 0$. [Sol: $\sin\delta_0 \simeq -\beta(k/\mu)[1 - (k^2/\mu^2)]$, β/μ, $(\beta - 1/\beta)/\mu$, with $\beta = 2mV_0/(\hbar^2\mu^2)$].

25. Show that the scattering amplitude for low energy s-waves may be written as

$$f_0 = -\frac{\alpha}{1 + i\alpha k - \alpha k^2 r_{\text{eff}}/2},$$

    as well as in the form

$$f_0 = \frac{1}{k\cot\delta_0 - ik},$$

where α is the scattering length and r_{eff} is the effective range. Also verify that both versions of the amplitude satisfy the optical theorem.

26. The regular s-wave ($l = 0$) $u(r) \equiv u_{0,k}(r)$ function for elastic scattering of a spinless particle of mass m and linear momentum $\hbar\boldsymbol{k}$ by some central potential is

$$u(r) = \frac{\sin(kr)}{k} + \frac{b^2 - a^2}{b^2 + k^2} \frac{k\sinh(br)\cos(kr) - b\cosh(br)\sin(kr)}{k[b\cosh(br) + a\sinh(br)]}, \qquad (8)$$

where a and b are some constants and $k = |\boldsymbol{k}|$.

a) Calculate the phase shift. [Sol: $\tan\delta_0 = k(b-a)/(k^2 + ab)$].

b) Calculate the scattering length and the effective range. [Sol: $1/b - 1/a$, $2/(b-a)$].

Resonances

27. Consider a spinless particle of mass m, linear momentum $\hbar\boldsymbol{k}$, under the influence of the potential shell of value V_0 between $r = a$ and $r = b$ and zero otherwise. Calculate the s-wave phase shift and show that for large V_0 (with respect to what?) resonances occur approximately at energies which would be bound states if V_0 were infinite. Hints: For $l = 0$, the radial wave function for this potential is given by $R_0(r) = \sin(kr)/kr$ if $r < a$, $A\sin(Kr)/(Kr) - B\cos(Kr)/(Kr)$ if $a < r < b$ and $C\sin(kr + \delta_0)/(kr)$ if $r > b$, where $k = |\boldsymbol{k}|$, $E = \hbar^2 k^2/(2m)$, $E - V_0 = \hbar^2 K^2/(2m)$. Impose the continuity of the wave function and the first derivative at $r = a$ and of the logarithmic derivative at $r = b$. At energies corresponding to a box extending from $r = a$ to $r = b$ we have that $R_0(a) = R_0(b) = 0$. [Sol: $V_0 >> E$].

Inelastic scattering

28. Calculate the total cross section from a black disk of radius a at high energies. Compare your answer with the classical value and discuss the results. Hint: assume the values of l do not exceed a maximum value. [Sol: $2\pi a^2$; πa^2].

29. In an experiment, 650 MeV π^0 pions are scattered from a heavy, totally absorbing nucleus of radius 1.4 fm.

a) Estimate the total elastic and total inelastic cross sections. [Sol: 0.40 barn each].

b) Calculate the scattering amplitude $f(\theta)$ and check the validity of the optical theorem. [Sol: $f(\theta) = i/(2k)[1 + 3\cos\theta + (5/2)(3\cos^2 - 1) + (7/2)(5\cos^3\theta - 3\cos\theta)]$].

c) Using the scattering amplitude found in b), calculate and plot the differential cross section for elastic scattering. Calculate the total elastic cross section and verify that it agrees with the expression found in a).

Hint: the experiment can be viewed as a scattering of high-energy pions from a black disk.

30. Show that, for a given σ^l_{inel}, where $l\hbar$ is the orbital angular momentum related to the corresponding partial wave, i.e., $\sigma_{\text{inel}} = \sum_{l=0}^{\infty} \sigma^l_{\text{inel}}$, the elastic scattering cross section σ^l_{el}, where $\sigma_{\text{el}} = \sum_{l=0}^{\infty} \sigma^l_{\text{el}}$, is such that

$$\sqrt{\sigma^l_0} - \sqrt{\sigma^l_0 - \sigma^l_{\text{inel}}} \leq \sqrt{\sigma^l_{\text{el}}} \leq \sqrt{\sigma^l_0} + \sqrt{\sigma^l_0 + \sigma^l_{\text{inel}}},$$

where $\sigma^l_0 = \pi(2l+1)/k^2$.

The time-dependent approach

31. A spinless particle of momentum $\hbar\boldsymbol{k}$ is scattered by an harmonic time-dependent potential $\mathbb{V}(\boldsymbol{x}', t) = V(\boldsymbol{x}')\cos(\omega t)$, where ω is the angular frequency of the oscillations. Assume that the perturbation turns on at $t = 0$.

a) Show that at first order in the time-dependent perturbation theory [i.e., calculate $c_1(t)$], the energy of the scattered particle is increased or decreased by $\hbar\omega$.

b) Obtain $d\sigma/d\Omega$. [Sol: $[m^2/(4\pi^2\hbar^4 k)][k'_+ + k'_-]\left|\int_{-\infty}^{+\infty} d^3x' V(\boldsymbol{x}') e^{i\boldsymbol{K}\cdot\boldsymbol{x}'}\right|^2$, where $k'_\pm = \sqrt{k^2 \pm 2m\omega/\hbar}$, $\boldsymbol{K} = \boldsymbol{k} - \boldsymbol{k}'$].

Hints: To calculate $c_1(t)$, write $\cos(\omega t)$ as $(e^{i\omega t} + e^{-i\omega t})/2$ and show that the only appreciable contributions after long times come from

$\omega_{fi} \mp \omega = 0$, with $\omega_{fi} = (E_f - E_i)/\hbar$ or $E_f = E_i \pm \hbar\omega$ (you might find this calculation in several textbooks of quantum mechanics in the time-dependent perturbation theory section). This means $k = |\boldsymbol{k}| \neq k' = |\boldsymbol{k}'|$. When calculating $w_{i\to f}$ use the density $\rho(E_f)$, where E_f is written in terms of k', but when determining $d\sigma/d\Omega$, use the incident flux I_0 written in terms of k.